Eur. Phys. J. C 42, 17–24 (2005)
Digital Object Identifier (DOI) 10.1140/epjc/s2005-02251-8

THE EUROPEAN
PHYSICAL JOURNAL C

Relations between quark and lepton mixing angles and matrices

Nan Li[1], Bo-Qiang Ma[2,1,a]

[1] Department of Physics, Peking University, Beijing 100871, P.R. China
[2] CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, P.R. China

Received: 6 April 2005 /
Published online: 18 May 2005 –

Abstract. We discuss the relations between the mixing angles and the mixing matrices of quarks and leptons. With Raidal's numerical relations, we parameterize the lepton mixing (PMNS) matrix with the parameters of the quark mixing (CKM) matrix, and calculate the products of $V_{\mathrm{CKM}}U_{\mathrm{PMNS}}$ and $U_{\mathrm{PMNS}}V_{\mathrm{CKM}}$. Also, under the conjectures $V_{\mathrm{CKM}}U_{\mathrm{PMNS}} = U_{\mathrm{bimax}}$ or $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$, we get the PMNS matrix naturally, and test Raidal's relations in these two different versions. The similarities and the differences between the different versions are discussed in detail.

PACS. 14.60.Pq, 12.15.Ff

1 Introduction

The mixing of quarks and leptons is one of the fundamental problems in particle physics. But its origin is still unknown yet, and the mixing is described phenomenologically by the mixing matrices, i.e., the Cabibbo–Kobayashi–Maskawa (CKM) [1] matrix for quark mixing and the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) [2] matrix for lepton mixing. To understand the mixing problem, two aspects should be considered. One is the mixing matrix, and the other is the mixing angle. However, these mixing angles cannot be determined by the standard model (SM) itself but can only be fixed by the experimental data. So the mixing angles are taken as free parameters and are not correlated. Furthermore, the quark and lepton mixing matrices, which are composed of the mixing angles, are also independent of each other. If we can find the relation between these mixing angles or the relation between the mixing matrices, it will be helpful for our understanding of the inner essence of the SM and for model construction of the grand unified theory.

In this paper, we discuss the relations between the mixing angles and the mixing matrices of quarks and leptons, respectively. First, for the mixing angles, Raidal has suggested some numerical relations [3]:

$$
\begin{aligned}
&\theta_1^{\mathrm{CKM}} + \theta_1^{\mathrm{PMNS}}(\theta_{\mathrm{atm}}) = \frac{\pi}{4},\\
&\theta_2^{\mathrm{CKM}} \sim \theta_2^{\mathrm{PMNS}}(\theta_{\mathrm{chz}}) \sim \mathcal{O}(\lambda^3),\\
&\theta_3^{\mathrm{CKM}}(\theta_{\mathrm{C}}) + \theta_3^{\mathrm{PMNS}}(\theta_{\mathrm{sol}}) = \frac{\pi}{4},
\end{aligned}
\tag{1}
$$

where the θ_i are the mixing angles of the CKM and the PMNS matrices. With these relations, we can link the elements of the CKM and the PMNS matrices together, and then can express the CKM and the PMNS matrices in a unified way [4]. Furthermore, we can find the relation between these two mixing matrices.

Second, for the mixing matrices, we discuss the products of the CKM and the PMNS matrices. Both $V_{\mathrm{CKM}}U_{\mathrm{PMNS}}$ and $U_{\mathrm{PMNS}}V_{\mathrm{CKM}}$ are calculated in detail. We find that the product of the CKM and the PMNS matrices is rather near the bimaximal mixing pattern. So we can get the PMNS matrix in terms of the CKM matrix and the bimaximal mixing matrix. The PMNS matrix can be parameterized by the parameters of the CKM matrix, and the relations between the mixing angles are deduced naturally.

In Sects. 2 and 3, we discuss the quark and lepton mixing matrices, and the mixing angles and the parameterizations of quark and lepton mixing matrices, and show their relations. In Sect. 4, with the numerical relations between the quark and lepton mixing angles, we discuss the relation between the quark and lepton mixing matrices and point out the similarities and the differences of different versions. In Sect. 5, we discuss the relations between the mixing angles under the conjecture that the product of quark and lepton mixing matrices is the bimaximal mixing pattern. Some conclusions are given in Sect. 6.

[a] e-mail: mabq@phy.pku.edu.cn

2 The quark and lepton mixing matrices

To see the generation of the quark mixing matrix, let us consider the charge-changing weak current:

$$j = 2 \sum_{\alpha'=u',c',t'} \overline{u}_{\alpha'} \gamma_\rho d_{\alpha'}, \tag{2}$$

where the u-type and d-type quark fields $u_{\alpha'}$ and $d_{\alpha'}$ do not have definite masses but are linear combinations of the massive quark fields u_α and d_α,

$$u_{\alpha'} = \sum_{\alpha=u,c,t} V_u^{\alpha'\alpha} u_\alpha, \quad d_{\alpha'} = \sum_{\alpha=d,s,b} V_d^{\alpha'\beta} d_\beta, \tag{3}$$

where V_u and V_d are unitary matrices which can diagonalize the quark mass matrices. Substituting (3) into (2), we have

$$\begin{aligned} j &= 2 \sum_{\alpha',\alpha,\beta} \overline{u}_\alpha \gamma_\rho V_u^{\alpha\alpha'\dagger} V_d^{\alpha'\beta} d_\beta \\ &= 2 \sum_{\alpha,\beta} \overline{u}_\alpha \gamma_\rho V_{\mathrm{CKM}}^{\alpha\beta} d_\beta, \end{aligned}$$

where

$$V_{\mathrm{CKM}} = V_u^\dagger V_d. \tag{4}$$

V_{CKM} is the quark mixing (CKM) matrix, which links the flavor eigenstates to the mass eigenstates of quarks.

The CKM matrix is measured by different experiments to a good degree of accuracy [5], and the elements of the modulus of the CKM matrix are summarized as

$$\begin{pmatrix} 0.9739-0.9751 & 0.221-0.227 & 0.0029-0.0045 \\ 0.221-0.227 & 0.9730-0.9744 & 0.039-0.044 \\ 0.0048-0.014 & 0.037-0.043 & 0.9990-0.9992 \end{pmatrix}.$$

We can see that the CKM matrix is very near the unit matrix, and it can be parameterized by the Wolfenstein parameterization [6]:

$$V_{\mathrm{CKM}} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - \mathrm{i}\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-\mathrm{i}\eta) & -A\lambda^2 & 1 \end{pmatrix}, \tag{5}$$

where λ measures the strength of the deviation of V_{CKM} from the unit matrix ($\lambda = \sin\theta_{\mathrm{C}} = 0.2243 \pm 0.0016$, θ_{C} is the Cabibbo mixing angle), and A, ρ and η are the other three parameters, with the best fit values $A = 0.82$, $\rho = 0.20$ and $\eta = 0.33$ [5].

Similarly, the lepton mixing (PMNS) matrix can be written as

$$U_{\mathrm{PMNS}} = U_l^\dagger U_\nu, \tag{6}$$

where U_l and U_ν are unitary matrices, which can diagonalize the charged-lepton and the neutrino mass matrices, and U_{PMNS} links the flavor eigenstates to the mass eigenstates of leptons.

The elements of the modulus of the PMNS matrix are summarized as [7]

$$\begin{pmatrix} 0.77\text{–}0.88 & 0.47\text{–}0.61 & <0.20 \\ 0.19\text{–}0.52 & 0.42\text{–}0.73 & 0.58\text{–}0.82 \\ 0.20\text{–}0.53 & 0.44\text{–}0.74 & 0.56\text{–}0.81 \end{pmatrix}. \tag{7}$$

We can see that the PMNS matrix deviates from the unit matrix very much but is quite near the bimaximal mixing pattern, which reads

$$U_{\mathrm{bimax}} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -1/2 & 1/2 & \sqrt{2}/2 \\ 1/2 & -1/2 & \sqrt{2}/2 \end{pmatrix}. \tag{8}$$

Since the CKM matrix is quite near the unit matrix, and the PMNS matrix is quite near the bimaximal matrix, we may assume that the deviation of the PMNS matrix from bimaximal can just be described by the CKM matrix, that is

$$U_{\mathrm{PMNS}} V_{\mathrm{CKM}} = U_{\mathrm{bimax}}, \tag{9}$$

or

$$V_{\mathrm{CKM}} U_{\mathrm{PMNS}} = U_{\mathrm{bimax}}. \tag{10}$$

So we can get

$$U_{\mathrm{PMNS}} = U_{\mathrm{bimax}} V_{\mathrm{CKM}}^\dagger, \tag{11}$$

or

$$U_{\mathrm{PMNS}} = V_{\mathrm{CKM}}^\dagger U_{\mathrm{bimax}}. \tag{12}$$

Equations (9) and (10) have both been pointed out by Minakata and Smirnov [8], and the similar results have also been discussed in the literature [9]. Thus, the PMNS matrix can be expressed thoroughly by the CKM matrix and can be parameterized by the Wolfenstein parameters of the CKM matrix. So we can get the relations between the mixing angles of quarks and leptons. We will discuss these two cases in Sect. 5.

3 The mixing angles of the quark and lepton mixing matrices

Both the CKM matrix and the PMNS matrix can be written as

$$\begin{pmatrix} c_2 c_3 & c_2 s_3 & s_2 \mathrm{e}^{-\mathrm{i}\delta} \\ -c_1 s_3 - s_1 s_2 c_3 \mathrm{e}^{\mathrm{i}\delta} & c_1 c_3 - s_1 s_2 s_3 \mathrm{e}^{\mathrm{i}\delta} & s_1 c_2 \\ s_1 s_3 - c_1 s_2 c_3 \mathrm{e}^{\mathrm{i}\delta} & -s_1 c_3 - c_1 s_2 s_3 \mathrm{e}^{\mathrm{i}\delta} & c_1 c_2 \end{pmatrix}, \tag{13}$$

where $s_i = \sin\theta_i$, $c_i = \cos\theta_i$ (for $i = 1,2,3$), which describe the mixings between 2nd and 3rd, 3rd and 1st, and 1st and 2nd generations of quarks or leptons, and δ is the Dirac CP-violating phase. Altogether there are eight (four for quark sector and four for lepton sector) parameters in the mixing matrices, describing both the real and the imaginary parts of the mixing matrices. If neutrinos are of Majorana type, it is always feasible to parameterize the neutrino mixing matrix as a product of (13) and a diagonal phase matrix with two unremovable phase angles $\mathrm{diag}(1, \mathrm{e}^{\mathrm{i}\alpha}, \mathrm{e}^{\mathrm{i}\beta})$ [10], where α, β are the Majorana CP-violating phases.

For the quark sector, these angles have been measured to a good degree of accuracy (for example, see [5]). The best fit values of the three mixing angles are $\theta_1^{\mathrm{CKM}} = 2.4°$, $\theta_2^{\mathrm{CKM}} = 0.2°$, and $\theta_3^{\mathrm{CKM}}(\theta_{\mathrm{C}}) = 12.9°$.

For the lepton sector, with the help of various experimental data from the KamLAND [11], SNO [12], K2K [13], Super-Kamiokande [14] and CHOOZ [15] experiments, we now have a much better understanding of these mixing angles,

$$\begin{aligned}
&\sin^2 2\theta_{\mathrm{atm}} = 1.00 \pm 0.05,\\
&\sin^2 2\theta_{\mathrm{chz}} = 0 \pm 0.065,\\
&\tan^2 \theta_{\mathrm{sol}} = 0.41 \pm 0.05,
\end{aligned}$$

where θ_{atm}, θ_{chz}, and θ_{sol} are the mixing angles of atmospheric, CHOOZ and solar neutrino oscillations, and we have $\theta_{\mathrm{atm}} = \theta_1^{\mathrm{PMNS}} = 45.0° \pm 6.5°$, $\theta_{\mathrm{chz}} = \theta_2^{\mathrm{PMNS}} = 0° \pm 7.4°$ and $\theta_{\mathrm{sol}} = \theta_3^{\mathrm{PMNS}} = 32.6° \pm 1.6°$ [3].

An interesting numerical relation between the third mixing angles of quarks and leptons was pointed out by Smirnov [16]:

$$\theta_3^{\mathrm{CKM}}(\theta_{\mathrm{C}}) + \theta_3^{\mathrm{PMNS}}(\theta_{\mathrm{sol}}) = \frac{\pi}{4}. \tag{14}$$

This relation is called the quark–lepton complementarity (QLC) [8].

Raidal extended this relation to three generations [3]:

$$\begin{aligned}
&\theta_1^{\mathrm{CKM}} + \theta_1^{\mathrm{PMNS}}(\theta_{\mathrm{atm}}) = \frac{\pi}{4},\\
&\theta_2^{\mathrm{CKM}} \sim \theta_2^{\mathrm{PMNS}}(\theta_{\mathrm{chz}}) \sim \mathcal{O}(\lambda^3),\\
&\theta_3^{\mathrm{CKM}}(\theta_{\mathrm{C}}) + \theta_3^{\mathrm{PMNS}}(\theta_{\mathrm{sol}}) = \frac{\pi}{4}.
\end{aligned}$$

With these relations, we can find that the mixing angles of quarks and leptons are not independent of each other. And thus we can get the trigonometric functions of the mixing angles of leptons in terms of those of quarks, and link the parameters of the PMNS matrix with those of the CKM matrix. Therefore, we can parameterize the PMNS matrix and the CKM matrix in a same framework [4]. Then we can test the product relations in (9) and (10). We will discuss these cases in Sect. 4.

4 The relations between the mixing angles

In Wolfenstein parameterization of the CKM matrix, we have (to the order of λ^3)

$$\begin{aligned}
&\sin\theta_1^{\mathrm{CKM}} = A\lambda^2, \quad \cos\theta_1^{\mathrm{CKM}} = 1,\\
&\sin\theta_2^{\mathrm{CKM}}\mathrm{e}^{-\mathrm{i}\delta} = A\lambda^3(\rho - \mathrm{i}\eta), \quad \cos\theta_2^{\mathrm{CKM}} = 1,\\
&\sin\theta_3^{\mathrm{CKM}} = \lambda, \quad \cos\theta_3^{\mathrm{CKM}} = 1 - \frac{1}{2}\lambda^2.
\end{aligned} \tag{15}$$

Using (1), we can get the trigonometric functions of the mixing angles of leptons (to the order of λ^3)

$$\begin{aligned}
&\sin\theta_1^{\mathrm{PMNS}} = \sin\left(\frac{\pi}{4} - \theta_1^{\mathrm{CKM}}\right) = \frac{\sqrt{2}}{2}(1 - A\lambda^2),\\
&\cos\theta_1^{\mathrm{PMNS}} = \frac{\sqrt{2}}{2}(1 + A\lambda^2),\\
&\sin\theta_2^{\mathrm{PMNS}}\mathrm{e}^{-\mathrm{i}\delta} = A\lambda^3(\zeta - \mathrm{i}\xi),\\
&\cos\theta_2^{\mathrm{PMNS}} = 1,\\
&\sin\theta_3^{\mathrm{PMNS}} = \frac{\sqrt{2}}{2}\left(1 - \lambda - \frac{1}{2}\lambda^2\right),\\
&\cos\theta_3^{\mathrm{PMNS}} = \frac{\sqrt{2}}{2}\left(1 + \lambda - \frac{1}{2}\lambda^2\right),
\end{aligned} \tag{16}$$

where A and λ are the Wolfenstein parameters of the CKM matrix. So the CKM and the PMNS matrices have only one set of parameters with Raidal's numerical relations. Because there are in total four angles in the mixing matrix (three mixing angles and one CP-violating phase angle), and only two precise numerical relations are known, we have to introduce another two new parameters ζ and ξ to describe the PMNS matrix fully.

In (16), we set $\sin\theta_2^{\mathrm{PMNS}}\mathrm{e}^{-\mathrm{i}\delta} = A\lambda^3(\zeta - \mathrm{i}\xi)$. Because of the inaccurate experimental data of neutrino oscillations, we have not fixed the value of $|U_{e3}^{\mathrm{PMNS}}|$, and only its upper bound is known [7]. Therefore, we may also set $\sin\theta_2^{\mathrm{PMNS}}\mathrm{e}^{-\mathrm{i}\delta} = A\lambda^2(\zeta - \mathrm{i}\xi)$. The choice between them is to be determined by the future experimental data, and we discuss these two cases here, respectively.

Case 1. $\sin\theta_2^{\mathrm{PMNS}}\mathrm{e}^{-\mathrm{i}\delta} = A\lambda^3(\zeta - \mathrm{i}\xi)$.

Substituting (16) into (13), we can get the PMNS matrix:

$$\begin{aligned}
U_{\mathrm{PMNS}}
&= \begin{pmatrix}
\frac{\sqrt{2}}{2}\left(1 + \lambda - \frac{1}{2}\lambda^2\right) & \frac{\sqrt{2}}{2}\left(1 - \lambda - \frac{1}{2}\lambda^2\right) & A\lambda^3(\zeta - \mathrm{i}\xi)\\
-\frac{1}{2}\left[1 - \lambda + \left(A - \frac{1}{2}\right)\lambda^2 - A\lambda^3(1 - \zeta - \mathrm{i}\xi)\right] & \frac{1}{2}\left[1 + \lambda + \left(A - \frac{1}{2}\right)\lambda^2 + A\lambda^3(1 - \zeta - \mathrm{i}\xi)\right] & \frac{\sqrt{2}}{2}(1 - A\lambda^2)\\
\frac{1}{2}\left[1 - \lambda - \left(A + \frac{1}{2}\right)\lambda^2 + A\lambda^3(1 - \zeta - \mathrm{i}\xi)\right] & -\frac{1}{2}\left[1 + \lambda - \left(A + \frac{1}{2}\right)\lambda^2 - A\lambda^3(1 - \zeta - \mathrm{i}\xi)\right] & \frac{\sqrt{2}}{2}(1 + A\lambda^2)
\end{pmatrix}\\
&= \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0\\
-\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2}\\
\frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2}
\end{pmatrix}
+ \lambda \begin{pmatrix}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0\\
\frac{1}{2} & \frac{1}{2} & 0\\
-\frac{1}{2} & -\frac{1}{2} & 0
\end{pmatrix}\\
&+ \lambda^2 \begin{pmatrix}
-\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0\\
-\frac{1}{2}\left(A - \frac{1}{2}\right) & \frac{1}{2}\left(A - \frac{1}{2}\right) & -\frac{\sqrt{2}}{2}A\\
-\frac{1}{2}\left(A + \frac{1}{2}\right) & \frac{1}{2}\left(A + \frac{1}{2}\right) & \frac{\sqrt{2}}{2}A
\end{pmatrix}
\end{aligned}$$

$$
+ \lambda^3 \begin{pmatrix} 0 & 0 & A(\zeta - \mathrm{i}\xi) \\ \frac{1}{2}A(1-\zeta-\mathrm{i}\xi) & \frac{1}{2}A(1-\zeta-\mathrm{i}\xi) & 0 \\ \frac{1}{2}A(1-\zeta-\mathrm{i}\xi) & \frac{1}{2}A(1-\zeta-\mathrm{i}\xi) & 0 \end{pmatrix} + \cdots \tag{17}
$$

We can see from (17) the followings.
(1) The bimaximal mixing pattern is deduced naturally as the leading-order approximation as long as we accept the numerical relations in (1).
(2) The leading and next-to-leading order terms are just the same as the expressions in the expansion of the PMNS matrix around the bimaximal mixing pattern by Rodejohann [17] and us [18].
(3) The Wolfenstein parameter λ can characterize both the deviation of the CKM matrix from the unit matrix (see (5)) and the deviation of the PMNS matrix from the exactly bimaximal mixing pattern (see the next-to-leading order term in (17)).

Since these two different kinds of deviations are characterized by only one parameter set, the product of the CKM matrix and the PMNS matrix may just be the exactly bimaximal mixing matrix ((9) and (10)). To see this clearly, we discuss these two versions of the product, respectively.
(i) $V_{\mathrm{CKM}}U_{\mathrm{PMNS}}$.
From (17) and (5), we have

$$
\begin{aligned}
V_{\mathrm{CKM}}U_{\mathrm{PMNS}} &= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{\sqrt{2}-1}{2} & -\frac{\sqrt{2}-1}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}-1}{2} & -\frac{\sqrt{2}-1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \\
&+ \lambda^2 \begin{pmatrix} -\frac{\sqrt{2}-1}{2} & -\frac{\sqrt{2}-1}{2} & 0 \\ -\frac{\sqrt{2}-1}{2} & \frac{\sqrt{2}-1}{2} & -\frac{\sqrt{2}}{4} \\ -\frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix} \\
&+ \lambda^3 \begin{pmatrix} -\frac{\sqrt{2}-1}{4} - \frac{1}{2}A(1-\rho+\mathrm{i}\eta) & \frac{\sqrt{2}-1}{4} + \frac{1}{2}A(1-\rho+\mathrm{i}\eta) & A\left[(\zeta-\mathrm{i}\xi) - \frac{\sqrt{2}}{2}(1-\rho+\mathrm{i}\eta)\right] \\ \frac{\sqrt{2}-1}{4} - \frac{1}{2}A(\zeta+\mathrm{i}\xi) & \frac{\sqrt{2}-1}{4} - \frac{1}{2}A(\zeta+\mathrm{i}\xi) & 0 \\ \frac{1}{2}A[\sqrt{2}(1-\rho+\mathrm{i}\eta) - (\zeta+\mathrm{i}\xi)] & \frac{1}{2}A[\sqrt{2}(1-\rho+\mathrm{i}\eta) - (\zeta+\mathrm{i}\xi)] & 0 \end{pmatrix} \\
&+ \cdots
\end{aligned} \tag{18}
$$

We can see from (18) that the deviation of the product of the CKM matrix and the PMNS matrix from the exactly bimaximal mixing matrix is of order λ.
(ii) $U_{\mathrm{PMNS}}V_{\mathrm{CKM}}$.

Similarly, we have

$$
\begin{aligned}
U_{\mathrm{PMNS}}V_{\mathrm{CKM}} &= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \\
&+ \lambda^2 \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}}{2}A \\ -\frac{1}{2}A & -\frac{1}{2}(\sqrt{2}-1)A & -\frac{1}{2}(\sqrt{2}-1)A \\ -\frac{1}{2}A & -\frac{1}{2}(\sqrt{2}-1)A & \frac{1}{2}(\sqrt{2}-1)A \end{pmatrix} \\
&+ \lambda^3 \begin{pmatrix} 0 & 0 & A\left[(\zeta-\mathrm{i}\xi) - \frac{\sqrt{2}}{2}(1-\rho+\mathrm{i}\eta)\right] \\ \frac{1}{2}A[\sqrt{2}(1-\rho-\mathrm{i}\eta) - (\zeta+\mathrm{i}\xi)] & -\frac{1}{2}A(\zeta+\mathrm{i}\xi) & \frac{1}{2}A(1-\rho+\mathrm{i}\eta) \\ \frac{1}{2}A[\sqrt{2}(1-\rho-\mathrm{i}\eta) - (\zeta+\mathrm{i}\xi)] & -\frac{1}{2}A(\zeta+\mathrm{i}\xi) & -\frac{1}{2}A(1-\rho+\mathrm{i}\eta) \end{pmatrix} \\
&+ \cdots
\end{aligned} \tag{19}
$$

We can see from (19) that the deviation of the product of the PMNS matrix and the CKM matrix from the exactly bimaximal mixing matrix is smaller (to the order of λ^2) than the former one. So the conjecture in (9) is better than the conjecture in (10).
Case 2. $\sin\theta_2^{\mathrm{PMNS}}\mathrm{e}^{-\mathrm{i}\delta} = A\lambda^2(\zeta - \mathrm{i}\xi)$.

Repeating this process, we get

$$
\begin{aligned}
U_{\mathrm{PMNS}} &= \begin{pmatrix} \frac{\sqrt{2}}{2}\left(1+\lambda-\frac{1}{2}\lambda^2\right) & \frac{\sqrt{2}}{2}\left(1-\lambda-\frac{1}{2}\lambda^2\right) & A\lambda^2(\zeta-\mathrm{i}\xi) \\ -\frac{1}{2}\left\{1-\lambda-\left[\frac{1}{2}-A(1+\zeta+\mathrm{i}\xi)\right]\lambda^2\right\} & \frac{1}{2}\left\{1+\lambda-\left[\frac{1}{2}-A(1-\zeta-\mathrm{i}\xi)\right]\lambda^2\right\} & \frac{\sqrt{2}}{2}(1-A\lambda^2) \\ \frac{1}{2}\left\{1-\lambda-\left[\frac{1}{2}+A(1+\zeta+\mathrm{i}\xi)\right]\lambda^2\right\} & -\frac{1}{2}\left\{1+\lambda-\left[\frac{1}{2}+A(1-\zeta-\mathrm{i}\xi)\right]\lambda^2\right\} & \frac{\sqrt{2}}{2}(1+A\lambda^2) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \\
&+ \lambda^2 \begin{pmatrix} -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & A(\zeta-\mathrm{i}\xi) \\ \frac{1}{2}\left[\frac{1}{2}-A(1+\zeta+\mathrm{i}\xi)\right] & -\frac{1}{2}\left[\frac{1}{2}-A(1-\zeta-\mathrm{i}\xi)\right] & -\frac{\sqrt{2}}{2}A \\ -\frac{1}{2}\left[\frac{1}{2}+A(1+\zeta+\mathrm{i}\xi)\right] & \frac{1}{2}\left[\frac{1}{2}+A(1-\zeta-\mathrm{i}\xi)\right] & \frac{\sqrt{2}}{2}A \end{pmatrix} \\
&+ \cdots
\end{aligned} \tag{20}
$$

Similarly, we have for
(i) $V_{\mathrm{CKM}}U_{\mathrm{PMNS}}$

$$
\begin{aligned}
&V_{\mathrm{CKM}}U_{\mathrm{PMNS}}\\
&=\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{\sqrt{2}-1}{2} & -\frac{\sqrt{2}-1}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}-1}{2} & -\frac{\sqrt{2}-1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}\\
&+\lambda^2 \begin{pmatrix} -\frac{\sqrt{2}-1}{2} & -\frac{\sqrt{2}-1}{2}A(\zeta-\mathrm{i}\xi) & A(\zeta-\mathrm{i}\xi) \\ -\frac{1}{2}[\sqrt{2}-1+A(\zeta+\mathrm{i}\xi)] & \frac{1}{2}[\sqrt{2}-1-A(\zeta+\mathrm{i}\xi)] & -\frac{\sqrt{2}}{4} \\ -\frac{1}{2}\left[\frac{1}{2}+A(\zeta+\mathrm{i}\xi)\right] & \frac{1}{2}\left[\frac{1}{2}-A(\zeta+\mathrm{i}\xi)\right] & 0 \end{pmatrix}\\
&+\cdots
\end{aligned}
\tag{21}
$$

and for
(ii) $U_{\mathrm{PMNS}}V_{\mathrm{CKM}}$

$$
\begin{aligned}
&U_{\mathrm{PMNS}}V_{\mathrm{CKM}}\\
&=\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\\
&+\lambda^2 \begin{pmatrix} 0 & 0 & A\left(\frac{\sqrt{2}}{2}+\zeta-\mathrm{i}\xi\right) \\ -\frac{1}{2}A(1+\zeta+\mathrm{i}\xi) & -\frac{1}{2}A(\sqrt{2}-1+\zeta+\mathrm{i}\xi) & -\frac{1}{2}(\sqrt{2}-1)A \\ -\frac{1}{2}A(1+\zeta+\mathrm{i}\xi) & -\frac{1}{2}A(\sqrt{2}-1+\zeta+\mathrm{i}\xi) & \frac{1}{2}(\sqrt{2}-1)A \end{pmatrix}\\
&+\cdots
\end{aligned}
\tag{22}
$$

Again, we find that the deviation of $U_{\mathrm{PMNS}}V_{\mathrm{CKM}}$ from the exactly bimaximal mixing matrix is rather small (to the order of λ^2), and that the deviation of $V_{\mathrm{CKM}}U_{\mathrm{PMNS}}$ from the exactly bimaximal mixing matrix is larger (to the order of λ). So the former conjecture in (9) is still better than the conjecture in (10).

In summary, in both the cases of $\sin\theta_2^{\mathrm{PMNS}}\mathrm{e}^{-\mathrm{i}\delta} = A\lambda^3(\zeta-\mathrm{i}\xi)$ and $\sin\theta_2^{\mathrm{PMNS}}\mathrm{e}^{-\mathrm{i}\delta} = A\lambda^2(\zeta-\mathrm{i}\xi)$, the product of $U_{\mathrm{PMNS}}V_{\mathrm{CKM}}$ is nearer to the exactly bimaximal mixing matrix than the product of $V_{\mathrm{CKM}}U_{\mathrm{PMNS}}$.

5 The relations between the mixing matrices

In the previous deductive process, we admit Raidal's numerical relations between the mixing angles of quarks and leptons beforehand and thus get the PMNS matrix in terms of the Wolfenstein parameters of the CKM matrix. Then we calculate the product of $V_{\mathrm{CKM}}U_{\mathrm{PMNS}}$ and $U_{\mathrm{PMNS}}V_{\mathrm{CKM}}$, and we compare their deviations from the exactly bimaximal mixing matrix. However, with the current experimental data, we can also make the conjectures $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$ or $V_{\mathrm{CKM}}U_{\mathrm{PMNS}} = U_{\mathrm{bimax}}$ at first, and then get the PMNS matrix straightforwardly. Thereafter we can find whether Raidal's relations hold good under these conjectures. We discuss the two different products, respectively. We have seen from Sect. 4 that $U_{\mathrm{PMNS}}V_{\mathrm{CKM}}$ is closer to the bimaximal mixing pattern (to the order of λ^2) than $V_{\mathrm{CKM}}U_{\mathrm{PMNS}}$ (to the order of λ), so this time we discuss the case $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$ first.

Case 1. $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$.

We suggest this product as a possibility for the relation between the quark and lepton mixing matrices. Although we have no theoretical fundamental reason for this suggestion, we can see that this product is consistent with (19) and (22) in Sect. 4. In the following deductive process, we can see that if we assume $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$, the QLC can be obtained directly and Raidal's relations can hold good, and the parameterization of the PMNS matrix can be deduced naturally.

Because V_{CKM} is unitary, we can get U_{PMNS} by multiplying $V_{\mathrm{CKM}}^{\dagger}$ on the right of U_{bimax},

$$
\begin{aligned}
&U_{\mathrm{PMNS}} = U_{\mathrm{bimax}}V_{\mathrm{CKM}}^{\dagger}\\
&=\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}\\
&+\lambda^2 \begin{pmatrix} -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2}A \\ \frac{1}{4} & \frac{\sqrt{2}}{2}A-\frac{1}{4} & -\frac{1}{2}A \\ -\frac{1}{4} & \frac{\sqrt{2}}{2}A+\frac{1}{4} & \frac{1}{2}A \end{pmatrix}\\
&+\lambda^3 \begin{pmatrix} 0 & 0 & \frac{\sqrt{2}}{2}A(1-\rho+\mathrm{i}\eta) \\ \frac{\sqrt{2}}{2}A(\rho+\mathrm{i}\eta) & 0 & -\frac{1}{2}A(1-\rho+\mathrm{i}\eta) \\ \frac{\sqrt{2}}{2}A(\rho+\mathrm{i}\eta) & 0 & \frac{1}{2}A(1-\rho+\mathrm{i}\eta) \end{pmatrix}\\
&+\cdots
\end{aligned}
\tag{23}
$$

We can see that the leading and the next-to-leading terms in (23) are just the same as those in (17) and (20). This indicates that Raidal's relations (see (1)) and $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$ are in very good consistency with each other.

To see this more clearly, we can calculate the trigonometric functions of the mixing angles of the PMNS matrix, and then calculate the sums of the corresponding angles of quarks and leptons.

From (23), we have

$$
\begin{aligned}
c_2^{\mathrm{PMNS}}s_3^{\mathrm{PMNS}} &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\lambda - \frac{\sqrt{2}}{4}\lambda^2,\\
c_2^{\mathrm{PMNS}}c_3^{\mathrm{PMNS}} &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\lambda - \frac{\sqrt{2}}{4}\lambda^2.
\end{aligned}
\tag{24}
$$

From (24) we have (to the order of λ^3)

$$\tan\theta_3^{\mathrm{PMNS}} = 1 - 2\lambda + 2\lambda^2 - 3\lambda^3. \tag{25}$$

Thus, we can get (to the order of λ^3)

$$s_3^{\mathrm{PMNS}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\lambda - \frac{\sqrt{2}}{4}\lambda^2,$$

$$c_3^{\mathrm{PMNS}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\lambda - \frac{\sqrt{2}}{4}\lambda^2. \tag{26}$$

Similarly, we have

$$\begin{aligned} s_1^{\mathrm{PMNS}} &= \frac{\sqrt{2}}{2} - A\lambda^2 + A\lambda^3, \\ c_1^{\mathrm{PMNS}} &= \frac{\sqrt{2}}{2} + A\lambda^2 - A\lambda^3. \end{aligned} \tag{27}$$

Also, we have

$$s_2^{\mathrm{PMNS}}\mathrm{e}^{-\mathrm{i}\delta} = -\frac{\sqrt{2}}{2}A\lambda^2 + \frac{\sqrt{2}}{2}(1-\rho+\mathrm{i}\eta)A\lambda^3, \tag{28}$$

and so

$$|s_2^{\mathrm{PMNS}}| = \frac{\sqrt{2}}{2}A\lambda^2\sqrt{(\lambda-\lambda\rho-1)^2+(\lambda\eta)^2}. \tag{29}$$

Substituting the best fit values of A, λ, ρ and η, we have

$$|s_2^{\mathrm{PMNS}}| = 0.48\lambda^2, \tag{30}$$

and $c_2^{\mathrm{PMNS}} = 1$ (to the order of λ^3).

Now we have got all the six trigonometric functions of the mixing angles of leptons, and we can calculate the sums of the mixing angles of quarks and leptons.

Using (15) and (26), we have

$$\begin{aligned} \sin(\theta_3^{\mathrm{CKM}} + \theta_3^{\mathrm{PMNS}}) &= s_3^{\mathrm{CKM}}c_3^{\mathrm{PMNS}} + c_3^{\mathrm{CKM}}s_3^{\mathrm{PMNS}} \\ &= \frac{\sqrt{2}}{2}, \end{aligned}$$

and thus

$$\theta_3^{\mathrm{CKM}} + \theta_3^{\mathrm{PMNS}} = \frac{\pi}{4}. \tag{31}$$

We can find that the QLC is satisfied precisely.

Similarly,

$$\sin(\theta_1^{\mathrm{CKM}} + \theta_1^{\mathrm{PMNS}}) = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2} - 1\right)A\lambda^2 + A\lambda^3,$$

and thus

$$\theta_1^{\mathrm{CKM}} + \theta_1^{\mathrm{PMNS}} = \frac{\pi}{4} - (\sqrt{2}-1)A\lambda^2 + \sqrt{2}A\lambda^3. \tag{32}$$

So Raidal's relation is violated a little (to the order of λ^2).

Also, for s_2^{PMNS}, we can find from (30) that $s_2^{\mathrm{PMNS}} \sim \lambda^2$; this differs from Raidal's relation slightly and is consistent with the parameterization in (20).

In summary, if we assume that $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$, we can get the PMNS matrix with the bimaxiaml matrix and the CKM matrix, all the elements of the PMNS matrix can be expressed by the parameters of the CKM matrix. The QLC is satisfied perfectly, and Raidal's relations can be deduced naturally (the deviation from Raidal's relations is of the order of λ^2).

Case 2. $V_{\mathrm{CKM}}U_{\mathrm{PMNS}} = U_{\mathrm{bimax}}$.

This relation has been pointed out by Giunti and Tanimoto [19] and discussed by some other authors [20,21]. Giunti and Tanimoto [19] suggested that the deviation of U_{PMNS} from the bimaximal mixing matrix is the CKM-like matrix, and Kang, Kim, and Lee [21] got this relation under the assumptions $Y_u = Y_d^{\mathrm{T}}$, $Y_u = Y_u^{\mathrm{T}}$ in SU(5) and $Y_\nu = Y_u$ in SO(10) grand unified theories.

Repeating the previous process, we can get the PMNS matrix as

$$\begin{aligned} U_{\mathrm{PMNS}} &= V_{\mathrm{CKM}}^{\dagger}U_{\mathrm{bimax}} \\ &= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &+ \lambda^2 \begin{pmatrix} -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 \\ \frac{1}{4} - \frac{1}{2}A & -\frac{1}{4} + \frac{1}{2}A & -\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2}A \\ -\frac{1}{2}A & \frac{1}{2}A & \frac{\sqrt{2}}{2}A \end{pmatrix} \\ &+ \lambda^3 \begin{pmatrix} \frac{1}{2}A(1-\rho+\mathrm{i}\eta) & -\frac{1}{2}A(1-\rho+\mathrm{i}\eta) & \frac{\sqrt{2}}{2}A(1-\rho+\mathrm{i}\eta) \\ 0 & 0 & 0 \\ \frac{\sqrt{2}}{2}A(\rho+\mathrm{i}\eta) & \frac{\sqrt{2}}{2}A(\rho+\mathrm{i}\eta) & 0 \end{pmatrix} \\ &+ \cdots \end{aligned} \tag{33}$$

We can see that the leading term in (33) is the bimaximal mixing pattern like that in (17) and (20). However, from the next-to-leading term, there are differences between (33) and (17) and (20). This indicates that the degree of the breaking of Raidal's relations (see (1)) is larger than that of Case 1.

Similarly, we can get all the six trigonometric functions of the mixing angles of leptons.

From (33), we have (to the order of λ^3)

$$\begin{aligned} s_1^{\mathrm{PMNS}} &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(A + \frac{1}{4}\right)\lambda^2, \\ c_1^{\mathrm{PMNS}} &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(A + \frac{1}{4}\right)\lambda^2, \\ |s_2^{\mathrm{PMNS}}| &= \frac{\sqrt{2}}{2}\lambda\sqrt{[A\lambda^2(1-\rho)-1]^2 + (A\lambda^2\eta)^2} \\ &= 0.68\lambda, \\ c_2^{\mathrm{PMNS}} &= 1 - 0.23\lambda^2, \\ s_3^{\mathrm{PMNS}} &= \frac{\sqrt{2}}{2} - \lambda - \frac{\sqrt{2}}{2}\lambda^2 + \left(A + \frac{1}{2}\right)\lambda^3, \\ c_3^{\mathrm{PMNS}} &= \frac{\sqrt{2}}{2} + \lambda - \frac{\sqrt{2}}{2}\lambda^2 - \left(A + \frac{1}{2}\right)\lambda^3. \end{aligned} \tag{34}$$

Also we can get the sums of mixing angles of quarks and leptons:

$$\sin(\theta_1^{\mathrm{CKM}} + \theta_1^{\mathrm{PMNS}}) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8}\lambda^2,$$

and thus

$$\theta_1^{\mathrm{CKM}} + \theta_1^{\mathrm{PMNS}} = \frac{\pi}{4} - \frac{1}{4}\lambda^2. \quad (35)$$

And we have

$$\begin{aligned}&\sin(\theta_3^{\mathrm{CKM}} + \theta_3^{\mathrm{PMNS}})\\ &= \frac{\sqrt{2}}{2} + (\frac{\sqrt{2}}{2} - 1)\lambda\\ &\quad + \left(1 - \frac{3\sqrt{2}}{4}\right)\lambda^2 + \left(A + 1 - \frac{\sqrt{2}}{2}\right)\lambda^3,\end{aligned}$$

and thus

$$\begin{aligned}&\theta_1^{\mathrm{CKM}} + \theta_1^{\mathrm{PMNS}}\\ &= \frac{\pi}{4} - (\sqrt{2} - 1)\lambda\\ &\quad + \left(\sqrt{2} - \frac{3}{2}\right)\lambda^2 + (\sqrt{2}A + \sqrt{2} - 1)\lambda^3. \quad (36)\end{aligned}$$

We can see from (35) and (36) that both of the Raidal relations break down, and the QLC is broken to the order of λ. This breaking has been pointed out by Minakata and Smirnov [8] and Kang, Kim, and Lee [21]. Comparing with (31) and (32), we can see that the difference is caused by the order of the product. If we set $V_{\mathrm{CKM}}U_{\mathrm{PMNS}} = U_{\mathrm{bimax}}$, the deviations from Raidal's relations are larger than the results if we set $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$.

Also, from (34), we know $|s_2^{\mathrm{PMNS}}| = 0.68\lambda$, so we can get $|U_{e3}^{\mathrm{PMNS}}| = 0.68\lambda$. Substituting the best fit value $\lambda = 0.2243$ [5] into it, we have

$$|U_{e3}^{\mathrm{PMNS}}| = 0.15.$$

This value is quite near the upper bound of $|U_{e3}^{\mathrm{PMNS}}| < 0.20$. However, from (30), we know $|s_2^{\mathrm{PMNS}}| = 0.48\lambda^2$, so we can get

$$|U_{e3}^{\mathrm{PMNS}}| = 0.48\lambda^2 = 0.024.$$

We can see that this result is more consistent with the current experimental upper bound.

From the discussions above, we can see the non-equivalence of (9) or (10) and Raidal's numerical relations of the mixing angles, which means that we cannot get Raidal's numerical relations of the mixing angles *exactly* from (9) or (10), and vice versa. There are small deviations from the *exact* Raidal numerical relations of the mixing angles if we take (9) or (10) as precise results. (For example, see (32) and (36).)

Furthermore, we find that the product $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$ is better than $V_{\mathrm{CKM}}U_{\mathrm{PMNS}} = U_{\mathrm{bimax}}$ from the viewpoints of both symmetry and phenomenological considerations. Of course, if the deviation of the PMNS matrix from the bimaximal mixing matrix is not exactly the CKM matrix but is just the CKM-like matrix [19–21] (i.e., the elements of the matrix have the same hierarchy as the Wolfenstein parameterization, but with not exactly the same Wolfenstein parameters), (10) may still be satisfied. The two different cases can be further discriminated by future experiments.

If the relation $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$ is supported by the future experimental data, using (4) and (6) we have

$$U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_l^\dagger U_\nu V_u^\dagger V_d = U_{\mathrm{bimax}}.$$

However, we know that U_l, U_ν, V_u and V_d are not definite, and we can set U_l and V_d to be the unit matrix by redefining the quark and lepton fields, and thus we have

$$U_{\mathrm{PMNS}} = U_\nu, \quad V_{\mathrm{CKM}} = V_u^\dagger,$$

and thus

$$U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_\nu V_u^\dagger = U_{\mathrm{bimax}}.$$

So we can find that the relation between the CKM and the PMNS matrices can be transformed to the relation between V_u and U_ν, and we may regard this as the complementarity of quark and lepton mixing matrices.

6 Conclusions

In this paper, we explore the relations between the mixing angles and mixing matrices of quarks and leptons. For the mixing angles, with Raidal's relations, we can link the mixing angles of quarks and leptons in a same framework and then express their mixing matrices in a unified way, i.e., we can parameterize the PMNS matrix with the Wolfenstein parameters of the CKM matrix [4]. With this unified parameterization, we discuss the relations between the quark and lepton mixing matrices. Both $V_{\mathrm{CKM}}U_{\mathrm{PMNS}}$ and $U_{\mathrm{PMNS}}V_{\mathrm{CKM}}$ are calculated in detail, and we can find that $U_{\mathrm{PMNS}}V_{\mathrm{CKM}}$ is closer to the bimaximal mixing matrix than $V_{\mathrm{CKM}}U_{\mathrm{PMNS}}$.

Similarly, for the relation between the quark and lepton mixing matrices, if we have $V_{\mathrm{CKM}}U_{\mathrm{PMNS}} = U_{\mathrm{bimax}}$, we can find that Raidal's relations will violate; especially the elegant quark–lepton complementarity (QLC) will break down (the degree of breaking is of order λ). On the contrary, if we set $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$, we can see that Raidal's relations will hold good to the order of λ^2, and the QLC will be a precise relation exactly. Although $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$ is still a phenomenological suggestion, it is consistent with the experimental data and is supported by the analysis in Sect. 4. Future experimental discrimination between the two different cases of $V_{\mathrm{CKM}}U_{\mathrm{PMNS}} = U_{\mathrm{bimax}}$ or $U_{\mathrm{PMNS}}V_{\mathrm{CKM}} = U_{\mathrm{bimax}}$ will shed light on our understanding of the relation between the quark and lepton mixing matrices, and will be also helpful for the future model construction of the quark and lepton mixing matrices in a grand unified theory.

Acknowledgements. We are grateful to Prof. Xiao-Gang He for his stimulating suggestions, and to Feng Huang for some discussion. This work is partially supported by National Natural Science Foundation of China (Nos. 10025523, 90103007, and 10421003) and by the Key Grant Project of Chinese Ministry of Education (No. 305001).

References

1. N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi, T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973)
2. B. Pontecorvo, Sov. Phys. JETP **6**, 429 (1958); Z. Maki, M. Nakagawa, S. Sakata, Prog. Theor. Phys. **28**, 870 (1962)
3. M. Raidal, Phys. Rev. Lett. **93**, 161801 (2004)
4. N. Li, B.-Q. Ma, Phys. Rev. D **71**, 097301 (2005)
5. Particle Data Group, S. Eidelman et al., Phys. Lett. B **592**, 1 (2004)
6. L. Wolfenstein, Phys. Rev. Lett. **51**, 1945 (1983)
7. M.C. Gonzalez-Garcia, hep-ph/0410030
8. H. Minakata, A.Yu. Smirnov, Phys. Rev. D **70**, 073009 (2004)
9. G. Altarelli, F. Feruglio, I. Masina, Nucl. Phys. B **689**, 157 (2004); A. Romanino, Phys. Rev. D **70**, 013003 (2004); C.A. de S. Pires, J. Phys. G **30**, B29 (2004); S.T. Petcov, W. Rodejohann, Phys. Rev. D **71**, 073002 (2005); J. Ferrandis, S. Pakvasa, Phys. Lett. B **603**, 184 (2004)
10. J. Schechter, J.W.F. Valle, Phys. Rev. D **22**, 2227 (1980); S.M. Bilenky, J. Hosek, S.T. Petcov, Phys. Lett. B **94**, 495 (1980)
11. KamLAND Collaboration, K. Eguchi et al., Phys. Rev. Lett. **90**, 021802 (2003)
12. SNO Collaboration, S.N. Ahmed et al., Phys. Rev. Lett. **92**, 181301 (2004)
13. K2K Collaboration, M.H. Ahn et al., Phys. Rev. Lett. **90**, 041801 (2003)
14. Super-Kamiokande Collaboration, Y. Fukuda et al., Phys. Rev. Lett. **81**, 1562 (1998); Y. Ashie et al., Phys. Rev. Lett. **93**, 101801 (2004); C.K. Jung, C. McGrew, T. Kajita, T. Mann, Anna. Rev. Nucl. Part. Sci. **51**, 451 (2001)
15. CHOOZ Collaboration, M. Apollonio et al., Phys. Lett. B **420**, 397 (1998); Palo Verde Collaboration, F. Boehm et al., Phys. Rev. Lett. **84**, 3764 (2000)
16. A.Yu. Smirnov, hep-ph/0402264
17. W. Rodejohann, Phys. Rev. D **69**, 033005 (2004)
18. N. Li, B.-Q. Ma, Phys. Lett. B **600**, 248 (2004)
19. C. Giunti, M. Tanimoto, Phys. Rev. D **66**, 053013 (2002); D **66**, 113006 (2002)
20. P.H. Frampton, S.T. Petcov, W. Rodejohann, Nucl. Phys. B **687**, 31 (2004)
21. S.K. Kang, C.S. Kim, J. Lee, hep-ph/0501029